Forest Science, Vol. 34, No. 1, pp. 243–249.

Determining Investment-Efficient Diameter Distributions for Uneven-Aged Northern Hardwoods

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ABSTRACT. The constant "q" of de Liocourt is often used to model the diameter distribution of uneven-aged forest stands and produces "balanced" size class distributions. Claims that such distributions are the most efficient for uneven-aged management or necessary to achieve sustention of production have led to the use of the concept in a variety of management models. Using a northern hardwood mixed-species growth model and a Weibull distribution function, we demonstrate that "balanced" diameter distributions are not investment-efficient and are not necessary to achieve sustention of production over a fixed cutting cycle. Results are very sensitive to per tree price and maximum tree size assumptions. FOR. SCI. 34(1):243–249.

ADDITIONAL KEY WORDS. Balanced size class distributions, Weibull distribution, nonlinear programming, land expectation value, optimal harvesting.

BALANCED DIAMETER DISTRIBUTIONS associated with the residual growing stock in uneven-aged forest stands are often characterized by de Liocourt's (1898) constant "q." Leak (1964) and Meyer (1952) define a balanced uneven-aged forest as "one that can produce a sustained yield while maintaining an essentially constant structure and volume." A balanced diameter distribution is expected to show a smooth geometric progression of the number of trees in successive diameter classes, with the ratio of the number of trees in a given diameter class to those in the next larger class defined as "q." Tests of this theory in the United States have produced mixed results (Pro: Meyer and Stevenson 1943; Schmelz and Lindsey 1965; Con: Murphy and Farrar 1981; Hanley et al. 1975). Nevertheless, a variety of management models are based on the assumption of a constant "q" (e.g., Hansen 1984, Solomon et al. 1986, Graham and Smith 1983, Alexander and Edminster 1977, Lyon 1983, Hall and Bruna 1983).

Leak (1965), Meyer (1952, 1953), Murphy and Farrar (1982), and Moser (1976) described the relationship between the exponential distribution function and the constant "q." Subsequently, Hyink and Moser (1983), Stiff (1979), Martin (1982) and Bare and Opalach (1987a, 1987b) used the Weibull distribution function to describe the diameter distribution of uneven-aged forest stands. Because the exponential distribution function can be obtained as a special case of the Weibull (i.e., when the shape parameter of the Weibull is set equal to 1), the use of the Weibull allows one to test the assumptions that a balanced diameter distribution is most efficient for uneven-aged management and is needed to achieve and maintain a sustainable equilibrium solution.

MODELING WITH THE WEIBULL

Martin (1982) used Adams and Ek's (1974) northern hardwood mixed-species growth model in conjunction with a Weibull distribution function to determine optimal investment-efficient sustainable equilibrium diameter distributions. Such distributions maximize value growth over a fixed cutting cycle for a given value of residual growing stock. Further, the investment-efficient distribution that maximizes the land expectation value[1] is the preferred criterion if economic efficiency is the objective of management and a sustainable equilibrium solution is desired.

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[1] Land expectation value is the present value of the difference between periodic value growth and the opportunity cost of holding land and timber over an infinite series of identical cutting cycles.

Reprinted from the Forest Science, Vol. 34, No. 1, March 1988

TABLE 1. Martin's investment-efficient diameter distribution for northern hardwoods—good site. (interest = 5%; cutting cycle = 5 yr; maximum tree = 23 in.)

Diameter class (in.)	Trees/ac (residual)	Price per tree ($)
6	126.6	0.14
8	46.5	0.34
10	17.1	0.60
12	6.3	4.23
14	2.3	6.77
16	0.8	9.45
18	0.3	12.31
20	0.1	16.05
22	0.0	19.46

Weibull Parameters: shape (c) = 1.0, scale (b) = 1.999

Basal area = 59.9 ft^2
q = 2.72
Sawtimber growth = 846 bf
Sawtimber stocking = 1087 bf
Trees/ac = 200.1
Pulpwood growth = 2.50 cords
Pulpwood stocking = 10.87 cords

Value growth = $53.99
Land expectation value = $95
Value of Growing Stock = $100

Note: In all tables involving the Weibull distribution, residual trees per acre are reported as 0.0 if they occur with a frequency less than 0.1 trees per acre. However, the Weibull distribution function assigns a nonzero probability to all tree diameters up to, and including, the maximum tree size specified.
Source: Martin (1982)

Using the Weibull distribution function, Martin characterized the residual diameter distribution in terms of three decision variables: the two parameters of the Weibull distribution and the number of trees. Not only did this reduce the dimensions of the decision space of the optimization problem, it also facilitated the testing of the assumption of a constant progression of the number of trees in successive diameter classes. The mathematical formulation of the model is available elsewhere and is not repeated here (e.g., Martin 1982, Bare and Opalach 1987a, 1987b). In addition, Haight et al. (1985) and Haight (1985) provide useful insights regarding optimal conversion strategies using the same growth model.

Based on the Adams and Ek (1974) growth model, Martin (1982) found that the shape parameter (c) of the Weibull distribution function that maximized the land expectation value was sufficiently close to 1—implying the occurrence of a balanced sustainable equilibrium diameter distribution for the residual stand. Thus, he reported that use of the Weibull distribution function resulted in balanced diameter distributions—at least for the growth model being used. Earlier, Adams (1976) used a diameter class optimization algorithm and found that the optimal investment-efficient diameter distribution that maximized land expectation value was not characterized by a constant "q." In an effort to clarify this apparent discrepancy, the following analysis was undertaken.

ANALYSIS AND DISCUSSION

Table 1 presents the optimal sustainable equilibrium investment-efficient diameter distribution for an uneven-aged stand obtained using Adams and Ek's (1974) northern hardwood growth model as reported by Martin (1982). For this, and other results reported herein, a five-year cutting cycle, 5% real rate of interest, and good site land (i.e., average site index = 70) are assumed, and the optimal solution is the one that maximizes the land expectation value.

Martin's (1982) results in Table 1 assume that the Weibull shape parameter (c) is equal to 1—thereby assuring a balanced residual diameter distribution.[2] In addition

[2] Martin (1982) originally did not constrain (c) to equal one, but after preliminary analysis decided that it was sufficiently close. Thus, to simplify numerical work, it was set equal to 1.

to this constraint, trees larger than 23 in. in diameter are not permitted in the residual stand. As discussed by Bare and Opalach (1987a, 1987b), when the Weibull distribution function is used to describe the diameter distribution of the residual stand, selection of a maximum tree size further constrains the Weibull parameters to assign some nonzero probability to all size classes up to, and including, the maximum diameter (See Table 1 note). Therefore, Martin (1982) presents constrained solutions to the investment-efficient diameter distribution problem. Per tree prices used in Martin's analysis are also shown in Table 1. Using 2 in. diameter classes, Martin's optimal diameter distribution is characterized by a "q" of 2.72.

Using the same northern hardwood growth model and assumptions listed above, but a different optimizing routine, an attempt was made to verify Martin's results. Whereas Martin used a gradient projection method, a direct search, derivative-free, constrained nonlinear programming algorithm—The Complex Method (Box 1965)—was used. As shown in Table 2, results replicate Martin's solution, although a few minor discrepancies are apparent. In addition, the number of trees to cut from each diameter class at the end of each cutting cycle is shown. The Weibull shape parameter (c) is constrained to equal 1 in this verification of Martin's (1982) results and trees up to, and including, the maximum diameter of 23 in. are present (see Table 2 note).

Next, the constraint of a constant "q" (i.e., a Weibull distribution with shape parameter (c) equal to 1) was removed, and the steady-state equilibrium problem was resolved. With (c) unconstrained, the results, in Table 3 are superior to those in Table 2. Thus, relaxing the constraint of maintaining a balanced steady-state diameter distribution at the end of each cutting cycle leads to the attainment of a superior sustainable equilibrium solution.

This result illustrates that constrained solutions are inferior to unconstrained solutions, and calls into question the wisdom of constraining (c) to equal 1. Results shown in Tables 1–3 presume that no tree larger than 23 in. in diameter is present in the residual stand. Furthermore, the Weibull distribution assumption assigns non-

TABLE 2. Verification of Martin's investment-efficient diameter distribution for northern hardwoods—good site. (interest = 5%; cutting cycle = 5 yr.; maximum tree = 23 in.)

Diameter class (in.)	Trees/ac (residual)	Trees/ac (cut)	Price per tree ($)
6	127.8	0.0	0.14
8	46.5	15.8	0.34
10	16.9	7.9	0.60
12	6.1	3.7	4.23
14	2.2	1.7	6.77
16	0.8	0.8	9.45
18	0.3	0.3	12.31
20	0.1	0.1	16.05
22	0.0	0.1	19.46

Weibull Parameters: shape (c) = 1.0, scale (b) = 1.977

Basal area = 59.7 ft^2
q = 2.75
Sawtimber growth = 846 bf
Sawtimber stocking = 1097 bf

Trees/ac = 200.7
Pulpwood growth = 2.53 cords
Pulpwood stocking = 10.88 cords

Value growth = $53.34
Land expectation value = $94.39

Value of growing stock = $98.69

Note: Minor differences in results shown in Tables 1 and 2 are due largely to the optimization algorithms used.

Note: In all tables involving the Weibull distribution, residual trees per acre are reported as 0.0 if they occur with a frequency less than 0.1 trees per acre. However, the Weibull distribution function assigns a nonzero probability to all tree diameters up to, and including, the maximum tree size specified.

TABLE 3. Optimal investment-efficient diameter distribution for northern hardwoods—good site. (interest = 5%; cutting cycle = 5 yr; maximum tree = 23 in.)

Diameter class (in.)	Trees/ac (residual)	Trees/ac (cut)	Price per tree ($)
6	118.3	0.0	0.14
8	86.6	0.2	0.34
10	32.3	12.4	0.60
12	8.7	7.5	4.23
14	1.8	2.9	6.77
16	0.3	0.9	9.45
18	0.0	0.2	12.31
20	0.0	0.0	16.05
22	0.0	0.0	19.46

Weibull Parameters: shape (c) = 1.43, scale (b) = 2.708

Basal area = 80.4 ft^2 Trees/ac = 248.2
Pulpwood growth = 1.88 cords Pulpwood stocking = 16.30 cords
Sawtimber growth = 1271 bf Sawtimber stocking = 1103 bf

Value growth = $70.47 Value of growing stock = $118.36
Land expectation value = $136.71

Note. In all tables involving the Weibull distribution, residual trees per acre are reported as 0.0 if they occur with a frequency less than 0.1 trees per acre. However, the Weibull distribution function assigns a nonzero probability to all tree diameters up to, and including, the maximum tree size specified.

zero probabilities to all trees up to, and including, 23 in. in diameter. However, for the unconstrained case shown in Table 3, trees larger than 17 in. in diameter are effectively eliminated.[3]

The land expectation value associated with the solution shown in Table 3 is almost 50% greater than that shown in Table 2 where a constant "q" is presumed. This solution also results in an additional 20 ft^2 of basal area and 48 more trees in the residual stand. Lastly, sawtimber growth and residual stocking and pulpwood stocking are greater than in the Table 2 solution.

To test the assumption that the Weibull distribution function adequately characterizes uneven-aged northern hardwood stands, the Adams and Ek (1974) growth model was reoptimized using a diameter class approach. Although Adams (1976) previously reported investment-efficient solutions using a similar approach, results reported here cannot be compared to his because of site quality and per-tree price differences. These results provide the unconstrained optimum in that no binding maximum tree size constraints or requirements to achieve a balanced diameter distribution are present.

As noted in Table 4, the use of a diameter class model does not require that trees be present over the entire diameter class range as when the Weibull distribution function is assumed. Clearly, the investment-efficient solution shown in Table 4 is superior to the more highly constrained solution shown in Table 3 as the land expectation value is over 85% greater. The other striking difference is that no trees larger than 13 in. in diameter are carried in the residual stand, although more trees per acre are present. This clearly shows the significance of maximum tree size requirements when the Weibull distribution function is assumed to describe the residual diameter distribution. Lastly, the solution shown in Table 4 does not exhibit characteristics of a balanced diameter distribution.

Table 5 contains the optimal investment-efficient diameter distribution obtained

[3] Although trees larger than 17 in. in diameter are present in the residual diameter distribution, they do not occur with a frequency greater than 0.1 trees/ac, and are reported as 0.0 in this and all tables involving the Weibull distribution.

TABLE 4. Reoptimization of Adams and Ek (1974) diameter class model for northern hardwoods—good site. (interest = 5%; cutting cycle = 5 yr; maximum tree = 23 in.)

Diameter class (in.)	Trees/ac (residual)	Trees/ac (cut)	Price per tree ($)
6	117.0	0.0	0.14
8	86.0	0.0	0.34
10	66.6	0.0	0.60
12	0.1	22.1	4.23
14	0.0	0.0	6.77
16	0.0	0.0	9.45
18	0.0	0.0	12.31
20	0.0	0.0	16.05
22	0.0	0.0	19.46

Basal area = 89.5 ft^2
Pulpwood growth = 0.0 cords
Sawtimber growth = 2040 bf

Trees/ac = 269.7
Pulpwood stocking = 21.35 cords
Sawtimber stocking = 15 bf

Value growth = $93.94
Land expectation value = $253.65

Value of growing Stock = $86.36

Note: Although trees up to, and including, 23 inches were permitted, no trees larger than 13 inches were present in the optimal diameter class solution.

using the Weibull distribution function when the maximum tree size is restricted to 11 in. in diameter and (c) is not constrained to equal 1. These limits were imposed to determine if the Weibull distribution function could adequately describe the optimal diameter distribution produced by the diameter class optimization model (Table 4). A previous attempt using a maximum tree size limitation of 13 in. in diameter failed to replicate the diameter-class results and produced a land expectation value of only $145.15/ac. The results shown in Table 5 illustrate that an 11 in. diameter maximum tree size limitation very closely replicates the Table 4 solution. However, this may be

TABLE 5. Optimal investment-efficient diameter distribution for northern hardwoods—good site. (interest = 5%; cutting cycle = 5 yr; maximum tree = 11 in.)

Diameter class (in.)	Trees/ac (residual)	Trees/ac (cut)	Price per tree ($)
6	117.2	0.0	0.14
8	86.1	0.0	0.34
10	66.7	0.0	0.60
12	0.0	22.2	4.23
14	0.0	0.0	6.77
16	0.0	0.0	9.45
18	0.0	0.0	12.31
20	0.0	0.0	16.05
22	0.0	0.0	19.46

Weibull Parameters: shape (c) = 0.952, scale (b) = 8.628

Basal area = 89.5 ft^2
Pulpwood growth = 0.0 cords
Sawtimber growth = 2039 bf

Trees/ac = 270.0
Pulpwood stocking = 21.38 cords
Sawtimber stocking = 0 bf

Value growth = $93.77
Land expectation value = $253.69

Value of growing stock = $85.71

Note: In all tables involving the Weibull distribution, residual trees per acre are reported as 0.0 if they occur with a frequency less than 0.1 trees per acre. However, the Weibull distribution function assigns a nonzero probability to all tree diameters up to, and including, the maximum tree size specified.

largely due to the correspondence of the three-dimensional decision space within which the optimization occurs and the three diameter classes to which trees can be assigned. Further, it is not obvious that an analyst intent on using the Weibull distribution function would specify an 11 in. diameter maximum tree size limitation on *a priori* grounds. Thus, extreme care should be taken when using the Weibull distribution function to derive investment-efficient sustainable equilibrium diameter distributions for uneven-aged stands.

Lastly, all solutions presented in Tables 1–5 are highly dependent on per-tree price, cutting cycle, and interest rate assumptions. Although all tree prices are held constant over time in real terms, they could be increased (decreased) to fit other expectations. Undoubtedly, the discontinuous nature of the per-tree price curve used by Martin (1982) is greatly influencing the solutions reported herein and may be partly responsible for the observed behavior when the Weibull distribution function is used.

CONCLUSIONS

Based on a reexamination of the optimization of the Adams and Ek (1974) growth model, as reported by Adams (1976) and Martin (1982), it is concluded that: (a) investment-efficient sustainable equilibrium diameter distributions for northern hardwoods using this growth model are not balanced, (b) if the Weibull distribution function is used to derive uneven-aged sustainable equilibrium diameter distributions, care must be taken to select appropriate maximum tree size limitations, and (c) maximum tree size and per-tree price assumptions play a crucial role when the Weibull distribution function is selected to describe the diameter distribution of uneven-aged stands.

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